Crossed modules of Bigroups and BiCrossed modules of Bigroups

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Abstract:

This paper is devoted to introduce the notions of crossed modules of bigroups and bicrossed modules of bigroups as a suitable generalization of the notions of crossed modules of groups and bicrossed modules of groups, respectively. This is done by embedding the categories of crossed modules of groups and bicrossed modules of groups in the category of crossed modules of bigroups and bicrossed modules of bigroups, respectively, (via isomorphism of categories). The notion of extended group and extended group homomorphism had also been introduced in the beginning of this paper. Finally, two adjoint pair of functors had been given between the category of bicrossed modules of bigroups and the category of bigroups.

Key words: Crossed modules; Bicrossed modules; Bigroups; Adjoint functors.

1. Introduction

Crossed modules of groups were introduced by Whitehead through his investigation of the algebraic structure of the second relative homotopy groups [3]. The notion of crossed modules of groups play an important role in homotopy theory, group presentations [7], algebraic K-theory [5] and homological algebra [1,4]. Recall that a crossed module of groups $(\mathrm{C},\mathrm{G},\partial,\theta)$ is a group homomorphism $\partial:\mathrm{C}\to\mathrm{G}$ together with an action $\theta:\mathrm{G}\times\mathrm{C}\to\mathrm{C}$ of G on C (usually written by $(\mathrm{g},\mathrm{c})={}^{\mathrm{g}}\mathrm{c}$) satisfying the following two conditions:

(CM1) ∂ is a precrossed module, i.e., $\partial({}^{\mathrm{g}}\mathrm{c})=\mathrm{g}\,\partial(\mathrm{c})\mathrm{g}^{-1}$, for all $\mathrm{c}\in\mathrm{C}$, $\mathrm{g}\in\mathrm{G}$.

(CM2) The Pieffer subgroup is trivial, i.e., ${}^{\partial(\mathrm{c}_2)}\mathrm{c}_1=\mathrm{c}_2\mathrm{c}_1{\mathrm{c}_2}^{-1}$, for all $\mathrm{c}_1,\mathrm{c}_2\in\mathrm{C}$.

When the action is unambiguous, we may write the crossed module of groups $(\mathrm{C},\mathrm{G},\partial,\theta)$ simply as $(\mathrm{C},\mathrm{G},\partial)$. A morphism of crossed modules of groups $(\mu,\eta):(\mathrm{C},\mathrm{G},\partial)\to(\mathrm{D},\mathrm{H},\delta)$ is a pair of group homomorphisms $\mu:\mathrm{C}\to\mathrm{D}$ and $\eta:\mathrm{G}\to\mathrm{H}$, such that $\delta\mu=\eta\,\partial$ and $\mu({}^{\mathrm{g}}\mathrm{c})={}^{\eta(\mathrm{g})}\mu(\mathrm{c})$, for all $\mathrm{c}\in\mathrm{C}$, $\mathrm{g}\in\mathrm{G}$. Crossed modules of groups and morphisms as defined above form a category, CModGrps. For more details, we refer the reader to Brown [6] and Baues [2]. Raad and Hana in [8] defined a bicrossed module of groups $(\mathrm{C},\mathrm{G},\mathrm{H},\partial,\delta)$ that is consist of two crossed modules of groups $(\mathrm{C},\mathrm{G},\partial)$ and $(\mathrm{C},\mathrm{H},\delta)$ such that the action of G on C and G on H are compatible, i.e. ${}^{{}^{\mathrm{g}}\mathrm{h}}\mathrm{c}={}^{{}^{\mathrm{h}}\mathrm{g}}\mathrm{c}$ for all $\mathrm{c}\in\mathrm{C}$, $\mathrm{g}\in\mathrm{G}$. They also defined a morphism $(\mu,\eta,\omega):(\mathrm{C},\mathrm{G},\mathrm{H},\partial,\delta)\to(\acute{\mathrm{C}},\acute{\mathrm{G}},\acute{\mathrm{H}},\acute{\partial},\acute{\delta})$ of bicrossed modules of groups ,bimorphism, that is a triple of group homomorphisms $\mu: C \to \acute{C}$, $\eta: G \to \acute{G}$ and $\omega: H \to \acute{H}$ such that $(\mu, \eta): (C, G, \partial) \to (\acute{C}, \acute{G}, \acute{\partial})$ and $(\mu, \omega): (C, H, \delta) \to (\acute{C}, \acute{H}, \acute{\delta})$ are morphisms of crossed modules of groups. Bicrossed modules of groups and morphisms as defined above form a category ,BiCModBiGrps, of bicrossed modules of bigroups**.** Vasantha Kandasamy in [9] defined the bigroup $(G, +, \cdot)$ that is a set with two binary operation '$+$' and '$\cdot$' such that there exist two proper subsets G_1 and G_2 of G satisfied the following conditions: (i) $G = G_1 \cup G_2$. (ii) $(G_1, +)$ is a group. (iii) $(G_2, \cdot)$ is a group. He also defined a bigroup homomorphism $\partial: C \to G$ is a map from bigroup $C = C_1 \cup C_2$ to $G = G_1 \cup G_2$ such that $\partial_1: C_1 \to G_1$ and $\partial_2: C_2 \to G_2$ are group homomorphisms where ∂_1 and ∂_2 are restriction of ∂ on C_1 and C_2 respectively, i.e. $\partial_1 = \partial / C_1$ and $\partial_2 = \partial / C_2$.

2. Extended groups

In the next section, we shall introduce and study the notion of a crossed module of bigroups (as generalization). To do this, it is required to introduce a definition of an extended group, which we will in following definition.

DEFINITION 2.1. Let $(C, *)$ be a group. The extended group by inner of C which is denoted by $C - \mathrm{Inn}(C)$ is a set including all elements of C and $\mathrm{Inn}(C)$, i.e. $C - \mathrm{Inn}(C) = C \cup \mathrm{Inn}(C)$. (for shortening in this paper, we will called it extended group).

Since, for any group C, $\mathrm{Inn}(C)$ with the composition relation"$\circ$" is a group, we can define the extended group homomorphism as follows;

DEFINITION 2.2. A map $\rho = \{\rho_1, \rho_2\}: C - \mathrm{Inn}(C) \to G - \mathrm{Inn}(G)$ of extended groups is called an extended group homomorphism if $\rho / C = \rho_1: C \to G$ is a group homomorphism and $\rho / \mathrm{Inn}(C) = \rho_2: \mathrm{Inn}(C) \to \mathrm{Inn}(G)$ is a group homomorphism.
Taking objects and morphisms as defined above, we obtain the category $\mathrm{Grps} - \mathrm{InnGrps}$.

PROPOSITION 2.3. The category of groups, Grps, and the category of extended groups, $\mathrm{Grps} - \mathrm{InnGrps}$, are isomorphic.

Proof:- define a functor $F: \mathrm{Grps} \longrightarrow \mathrm{Grps} - \mathrm{InnGrps}$ as follows:

(i) $F(C) = C - \mathrm{Inn}(C)$, for all $C \in \mathrm{obGrps}$, and

(ii) $F(\partial) = \{\rho_1, \rho_2\}$, for all $\partial \in \mathrm{Mor}_{\mathrm{Grps}}(C, G)$, where $\rho_1 = \partial: C \to G$ and $\rho_2 = \bar{\partial}: \mathrm{Inn}(C) \to \mathrm{Inn}(G)$ defined as $\rho_2(f_c) = f_{\partial(c)}$, for all $f_c \in \mathrm{Inn}(C)$. Since ∂ is a group homomorphism, then ρ_2 is a group homomorphism; $\rho_2(f_c f_{\acute{c}}) = \rho_2(f_{c\acute{c}}) = f_{\partial(c\acute{c})} = f_{\partial(c)\,\partial(\acute{c})} = f_{\partial(c)} f_{\partial(\acute{c})} = \rho_2(f_c)\rho_2(f_{\acute{c}})$, for all $f_c, f_{\acute{c}} \in \mathrm{Inn}(C)$, i.e. $\rho = \{\rho_1, \rho_2\}: C - \mathrm{Inn}(C) \to G - \mathrm{Inn}(G)$ is extended group homomorphism.

Also, define a functor $T: \mathrm{Grps} - \mathrm{InnGrps} \longrightarrow \mathrm{Grps}$ as follows:

(i) $T(C - \mathrm{Inn}(C)) = C$, for all $C - \mathrm{Inn}(C) \in \mathrm{obGrps} - \mathrm{InnGrps}$, and

(ii) $T(\rho = \{\rho_1, \rho_2\}) = \rho_1$, for all $\rho = \{\rho_1, \rho_2\} \in Mor_{Grps-InnGrps}\big(C - Inn(C), G - Inn(G)\big)$.

Therefore , it is easy to see that $TF = I_{Grps}$ and $FT = I_{Grps-InnGrps}$.

Hence $Grps \approx Grps - InnGrps$. ■

Note that the extended group $(C - Inn(C), *, \circ)$ with binary operations "$*$" and "$\circ$" (where $*$ which is defined on C and $\circ$ is the composition relation) is a bigroup, and therefore $Grps - InnGrps$ is subcategory of BiGrps. Then according to the proposition above, the category of groups, Grps, can be embedded in BiGrps as a subcategory.

3. Crossed Modules of Bigroups

According to proposition 2.3, we can there is an isomorphic between the category of crossed modules of groups, CModGrps, and the category of (which we shall call in this paper) crossed modules of extended groups, $CModGrps - InnGrps$, which we will give in the next proposition.

PROPOSITION 3.1. The category of crossed modules of groups, CModGrps, and the category of crossed modules of extended groups, $CModGrps - InnGrps$, are isomorphic.

Proof:- define a functor $F: CmodGrps \longrightarrow CmodGrps - InnGrps$ as follows:

(i) $F\big((C, G, \partial)\big) = (C - Inn(C), G - Inn(G), \rho = \{\rho_1, \rho_2\})$, for all $(C, G, \partial) \in$ obCmodGrps, where $\rho_1 = \partial: C \to G$ and $\rho_2 = \bar{\partial}: Inn(C) \to Inn(G)$. Note that, $Inn(G)$ has an action on $Inn(C)$ via the action of G on C as follows: ${f_g}_{f_c} = f_{g_c}$, for all $f_g \in Inn(G)$ and $f_c \in Inn(C)$. In fact $\rho_2: Inn(C) \to Inn(G)$ with the action above is a crossed modules of groups;

$$\text{(CM1) } \rho_2\left({f_g}_{f_c}\right) = \rho_2\left(f_{g_c}\right) = f_{\partial(g_c)} = f_{g\,\partial(c)g^{-1}} = f_g f_{\partial(c)} f_{g^{-1}} = f_g \rho_2(f_c) {f_g}^{-1}$$

, for all for all $f_g \in Inn(G)$ and $f_c \in Inn(C)$, and

(CM2) $\rho_2(f_c)_{f_{\acute{c}}} = f_{\partial(c)_{f_{\acute{c}}}} = f_{\partial(c)_{\acute{c}}} = f_{c\acute{c}c^{-1}} = f_c f_{\acute{c}} f_{c^{-1}} = f_c f_{\acute{c}} f_c^{-1}$, for all $f_c, f_{\acute{c}} \in Inn(C)$.

This implies that, the extended group homomorphism

$\rho = \{\rho_1, \rho_2\}: C - Inn(C) \to G - Inn(G)$ is a crossed modules of extended group i.e. $F\big((C, G, \partial)\big) = (C - Inn(C), G - Inn(G), \rho = \{\rho_1, \rho_2\}) \in obCmodGrps - InnGrps$.

(ii) $F(\mu, \eta) = \big((\mu, \bar{\mu}), (\eta, \bar{\eta})\big)$, for all $(\mu, \eta) \in Mor_{CmodGrps}\Big((C, G, \partial), (\acute{C}, \acute{G}, \acute{\partial})\Big)$.

Note that; $(\bar{\mu}, \bar{\eta}): \big(Inn(C), Inn(G), \bar{\partial}\big) \to \Big(Inn(\acute{C}), Inn(\acute{G}), \bar{\acute{\partial}}\Big)$ is morphism of crossed modules of groups;

$\bar{\mu}\left(f_{g_{f_c}}\right) = \bar{\mu}\left(f_{g_c}\right) = f_{\mu(g_c)} = f_{\eta(g)_{\mu(c)}} = f_{\eta(g)_{f_{\mu(c)}}} = \bar{\eta}(f_g)_{\bar{\mu}(f_c)}$, for all $f_g \in \mathrm{Inn}(G)$ and $f_c \in \mathrm{Inn}(C)$, and this shows that $((\mu,\bar{\mu}),(\eta,\bar{\eta}))$ is morphism of crossed modules of extended groups, i.e.

$$F(\mu,\eta) = ((\mu,\bar{\mu}),(\eta,\bar{\eta})) \in \mathrm{Mor}_{\mathrm{CmodGrps-InnGrps}}\left(F(C,G,\partial), F(\acute{C},\acute{G},\acute{\partial})\right).$$

Also, define a functor $T: \mathrm{CmodGrps-InnGrps} \longrightarrow \mathrm{CmodGrps}$ as follows:

(i) $T((C-\mathrm{Inn}(C), G-\mathrm{Inn}(G), \rho = \{\rho_1,\rho_2\})) = (C,G,\partial)$ for all $(C-\mathrm{Inn}(C), G-\mathrm{Inn}(G), \rho = \{\rho_1,\rho_2\}) \in \mathrm{obCmodGrps-InnGrps}$, and

(ii) $T((\mu,\bar{\mu}),(\eta,\bar{\eta})) = (\mu,\eta)$ for all $((\mu,\bar{\mu}),(\eta,\bar{\eta})) \in \mathrm{Mor}_{\mathrm{CmodGrps-InnGrps}}\left(F(C,G,\partial), F(\acute{C},\acute{G},\acute{\partial})\right)$.

Therefore, it is easy to see that $TF = I_{\mathrm{CmodGrps}}$ and $FT = I_{\mathrm{CmodGrps-InnGrps}}$.

Thus $\mathrm{CmodGrps} \approx \mathrm{CmodGrps-InnGrps}$. ■

The concept of crossed module of extended groups in the previous preposition can be seen as extended crossed module of groups , and which we can extend to a crossed module of bigroups as we shall show that in the beginning of the next definition.

DEFINITION 3.2. Let C and G be bigroups, where $C = C_1 \cup C_2$ and $G = G_1 \cup G_2$. A bigroup homomorphism $\partial : C \to G$ is called a crossed modules of bigroups denoted by (C,G,∂,θ) (or simply as (C,G,∂)), if $(C_1,G_1,\partial_1,\theta_1)$ and $(C_2,G_2,\partial_2,\theta_2)$ are a-crossed modules of groups, where $\partial_1 = \partial / C_1$ and $\partial_2 = \partial / C_2$. If, in the above definition, (C_1,G_1,∂_1) and (C_2,G_2,∂_2) are a precrossed module of groups, we call (C,G,∂) a precrossed module of bigroups.

DEFINITION 3.3. A morphism $(\mu,\eta): (C,G,\partial) \to (H,D,\delta)$ of crossed modules of bigroups is a pair of bigroup homomorphisms such that $(\mu_1,\eta_1): (C_1,G_1,\partial_1) \to (H_1,D_1,\delta_1)$ and $(\mu_2,\eta_2): (C_2,G_2,\partial_2) \to (H_2,D_2,\delta_2)$ are a morphism of crossed modules of groups where $\mu_1 = \mu / C_1$, $\mu_2 = \mu / C_2$, $\eta_1 = \eta / G_1$ and $\eta_2 = \partial / G_2$.

Taking objects and morphisms as defined above, we obtain the category CModBiGrps of crossed modules of bigroups.

Note that, $\mathrm{CModGrps-InnGrps} \subseteq \mathrm{CModBiGrps}$, and since $\mathrm{CModGrps} \approx \mathrm{CModGrps-InnGrps}$, we deduce that $\mathrm{CModGrps} \subseteq \mathrm{CModBiGrps}$, i.e., the category of crossed modules of groups is embedding (as a subcategory) in the category of crossed modules of bigroups (via isomorphisms of categories).

EXAMPLES 3.4. (1) Let N be a normal subbigroup of C. Then the inclusion map together with conjugations action is a crossed module of bigroups (N,C,i_N). Accordingly, any bigroup C may be regarded as a crossed module of bigroups in two ways via the identity map or the inclusion map, i.e., $(\mathrm{C}, \mathrm{C}, \mathrm{I_C})$ and $(1_\mathrm{C}, \mathrm{C}, \mathrm{i}_{1_\mathrm{C}})$ respectively, where 1_C denotes the trivial subbigroup.

(2) Any map of abelian bigroups $\partial: \mathrm{C} \to \mathrm{G}$ is a crossed module of bigroups with respect to the trivial action of G on C. Also, for any bigroup C, the inclusion map $\mathrm{i}: \mathrm{Cent(C)} \to \mathrm{C}$ with trivial action is crossed modules of bigroups.

4. BiCrossed Modules of Bigroups

According to the propositions (2.3) and (3.1), we can there is a new isomorphic between the category of bicrossed modules of groups, BiCModGrps, and the category of (which we shall call in this paper) bicrossed modules of extended groups, $\mathrm{BiCModGrps} - \mathrm{InnGrps}$, which we will give in the next proposition.

PROPOSITION 4.1. The category of bicrossed modules of groups, BiCModGrps, and the category of extended crossed modules of groups, $\mathrm{CModGrps} - \mathrm{InnGrps}$, are isomorphic.

Proof:- define a functor $\mathrm{F}: \mathrm{CmodGrps} \longrightarrow \mathrm{CmodGrps} - \mathrm{InnGrps}$ as follows:

(i) $\mathrm{F}((\mathrm{C}, \mathrm{G}, \mathrm{H}, \partial, \delta)) = (\mathrm{C} - \mathrm{Inn(C)}, \mathrm{G} - \mathrm{Inn(G)}, \mathrm{H} - \mathrm{Inn(H)}, \{\partial, \bar{\partial}\}, \{\delta, \bar{\delta}\})$ for all $(\mathrm{C}, \mathrm{G}, \mathrm{H}, \partial, \delta) \in \mathrm{obBiCmodGrps}$. Note that, from proposition (3.1) $(\mathrm{C} - \mathrm{Inn(C)}, \mathrm{G} - \mathrm{Inn(G)}, \{\partial, \bar{\partial}\})$ and $(\mathrm{C} - \mathrm{Inn(C)}, \mathrm{H} - \mathrm{Inn(H)}, \{\delta, \bar{\delta}\})$ are extended crossed modules of groups, and the crossed modules of groups $\bar{\partial}: \mathrm{Inn(C)} \to \mathrm{Inn(G)}$ with $\bar{\delta}: \mathrm{Inn(C)} \to \mathrm{Inn(H)}$ is a bicrossed module of groups $(\mathrm{Inn(C)}, \mathrm{Inn(G)}, \mathrm{Inn(H)}, \bar{\partial}, \bar{\delta})$, clearly is enough to proof this is show the compatible of the actions of $\mathrm{Inn(G)}$ and $\mathrm{Inn(H)}$ on $\mathrm{Inn(C)}$ and this satisfied:

$$\begin{aligned} \mathrm{f}_{\mathrm{g}_{\left(\mathrm{f}_{\mathrm{h}_{\mathrm{f_c}}}\right)}} = \mathrm{f}_{\mathrm{g}_{\left(\mathrm{f}_{\mathrm{h_c}}\right)}} &= \mathrm{f}_{\mathrm{g}_{\mathrm{h_c}}} \\ &= \mathrm{f}_{\mathrm{h}_{\mathrm{g_c}}} \quad (\text{since } (\mathrm{C}, \mathrm{G}, \mathrm{H}, \partial, \delta) \text{ is a bicrossed module}) \\ &= \mathrm{f}_{\mathrm{h}_{\left(\mathrm{f_{g_c}}\right)}} = \mathrm{f}_{\mathrm{h}_{\left(\mathrm{f}_{\mathrm{g}_{\mathrm{f_c}}}\right)}} \end{aligned}$$

For all for all $\mathrm{f_g} \in \mathrm{Inn(G)}$, $\mathrm{f_h} \in \mathrm{Inn(H)}$ and $\mathrm{f_c} \in \mathrm{Inn(C)}$.

This implies that, the extended group homomorphism

$\{\partial, \bar{\partial}\}: \mathrm{C} - \mathrm{Inn(C)} \to \mathrm{G} - \mathrm{Inn(G)}$ with $\{\delta, \bar{\delta}\}: \mathrm{C} - \mathrm{Inn(C)} \to \mathrm{H} - \mathrm{Inn(H)}$ is a bicrossed module of extended groups, $(\mathrm{C} - \mathrm{Inn(C)}, \mathrm{G} - \mathrm{Inn(G)}, \mathrm{H} - \mathrm{Inn(H)}, \{\partial, \bar{\partial}\}, \{\delta, \bar{\delta}\})$.
i.e. $\mathrm{F}((\mathrm{C}, \mathrm{G}, \mathrm{H}, \partial, \delta)) = (\mathrm{C} - \mathrm{Inn(C)}, \mathrm{G} - \mathrm{Inn(G)}, \mathrm{H} - \mathrm{Inn(H)}, \{\partial, \bar{\partial}\}, \{\delta, \bar{\delta}\}) \in$ $\mathrm{obBiCmodGrps} - \mathrm{InnGrps}$.

(ii)$\mathrm{F}(\mu, \eta, \omega) = ((\mu, \bar{\mu}), (\eta, \bar{\eta}), (\omega, \bar{\omega}))$,forall
$(\mu, \eta, \omega) \in \mathrm{Mor}_{\mathrm{BiCmodGrps}}\left((\mathrm{C}, \mathrm{G}, \mathrm{H}, \partial, \delta), (\acute{\mathrm{C}}, \acute{\mathrm{G}}, \acute{\mathrm{H}}, \acute{\partial}, \acute{\delta})\right)$. Note that, from proposition

(3.1) $(\bar{\mu},\bar{\eta})$ and $(\bar{\mu},\bar{\omega})$ are morphisms of extended crossed modules of groups, this implies that $(\bar{\mu},\bar{\eta},\bar{\omega})$ is a morphism of bicrossed modules of groups, therefore $\big((\mu,\bar{\mu}),(\eta,\bar{\eta}),(\omega,\bar{\omega})\big)$ is a morphism of bicrossed modules of extended groups. i.e. $F(\mu,\eta,\omega)=\big((\mu,\bar{\mu}),(\eta,\bar{\eta}),(\omega,\bar{\omega})\big)\in$ $\mathrm{Mor}_{\mathrm{BiCmodGrps-InnGrps}}\Big(F(C,G,H,\partial,\delta),F\big(\acute{C},\acute{G},\acute{H},\acute{\partial},\acute{\delta}\big)\Big)$.

Also, define a functor $T:\mathrm{BiCmodGrps-InnGrps}\longrightarrow\mathrm{BiCmodGrps}$ as follows:
(i) $T\Big(\big(C-\mathrm{Inn}(C),G-\mathrm{Inn}(G),H-\mathrm{Inn}(H),\{\partial,\bar{\partial}\},\{\delta,\bar{\delta}\}\big)\Big)=(C,G,H,\partial,\delta)$, for all $\big(C-\mathrm{Inn}(C),G-\mathrm{Inn}(G),H-\mathrm{Inn}(H),\{\partial,\bar{\partial}\},\{\delta,\bar{\delta}\}\big)\in\mathrm{obBiCmodGrps-InnGrps}$, and (ii)$T\big((\mu,\bar{\mu}),(\eta,\bar{\eta}),(\omega,\bar{\omega})\big)=(\mu,\eta,\omega)$forall $\big((\mu,\bar{\mu}),(\eta,\bar{\eta}),(\omega,\bar{\omega})\big)\in$ $\mathrm{Mor}_{\mathrm{BiCmodGrps-InnGrps}}\Big(\big(C-\mathrm{Inn}(C),G-\mathrm{Inn}(G),H-\mathrm{Inn}(H),\{\partial,\bar{\partial}\},\{\delta,\bar{\delta}\}\big),\Big(\acute{C}-\mathrm{Inn}\big(\acute{C}\big),\acute{G}-\mathrm{Inn}\big(\acute{G}\big),\acute{H}-\mathrm{Inn}\big(\acute{H}\big),\Big\{\acute{\partial},\bar{\acute{\partial}}\Big\},\Big\{\acute{\delta},\bar{\acute{\delta}}\Big\}\Big)\Big)$.

Therefore, it is easy to see that $TF=I_{\mathrm{BiCmodGrps}}$ and $FT=I_{\mathrm{BiCmodGrps-InnGrps}}$. Thus $\mathrm{BiCmodGrps}\approx\mathrm{BiCmodGrps-InnGrps}$. ■

The concept of bicrossed module of extended groups in the previous preposition can be seen as extended bicrossed module of groups , and which we can extend to a bicrossed module of bigroups as we shall show that in the beginning of the next definition.

DEFINITION 4.2. A bicrossed module of bigroups (C,G,H,∂,δ) consists of two crossed modules of bigroups (C,G,∂) and (C,H,δ) such that $(C_1,G_1,H_1,\partial_1,\delta_1)$ and $(C_2,G_2,H_2,\partial_2,\delta_2)$ are a bicrossed modules of groups.

DEFINITION 4.3. A morphism $(\mu,\eta,\omega):(C,G,H,\partial,\delta)\rightarrow\big(\acute{C},\acute{G},\acute{H},\acute{\partial},\acute{\delta}\big)$ of bicrossed modules of bigroups is a triple of bigroup homomorphisms $:C\rightarrow\acute{C}$, $\eta:G\rightarrow\acute{G}$ and $\omega:H\rightarrow\acute{H}$ such that $(\mu_1,\eta_1,\omega_1):(C_1,G_1,H_1,\partial_1,\delta_1)\rightarrow\big(\acute{C}_1,\acute{G}_1,\acute{H}_1,\acute{\partial}_1,\acute{\delta}_1\big)$ and $(\mu_2,\eta_2,\omega_2):(C_2,G_2,H_2,\partial_2,\delta_2)\rightarrow\big(\acute{C}_2,\acute{G}_2,\acute{H}_2,\acute{\partial}_2,\acute{\delta}_2\big)$ are morphisms of bicrossed modules of groups(bimorphisms).

Taking objects and morphisms as defined above, we obtain the category, BiCModBiGrps, of bicrossed modules of bigroups.
Note that, $\mathrm{BiCModGrps-InnGrps}\subseteq\mathrm{CModBiGrps}$, and since $\mathrm{BiCModGrps}\approx\mathrm{BiCModGrps-InnGrps}$, we deduce that $\mathrm{BiCModGrps}\subseteq\mathrm{BiCModBiGrps}$, i.e., the category of bicrossed modules of groups is embedding (as a subcategory) in the category of bicrossed modules of bigroups (via isomorphisms of categories).

EXAMPLES 4.4. (1) for any crossed module of bigroups (C,G,∂,θ), there is a bicrossed module of bigroups $\big(\mathrm{Cent}(C),G,C,\bar{\partial},i\big)$, where $\bar{\partial}:\mathrm{Cent}(C)\rightarrow G$ is defined as follows; $\bar{\partial}(c)=\begin{cases}\partial(c) & \text{if } \partial(c)\in\mathrm{Cent}(G)\\ 1_G & \text{e.w.}\end{cases}$, and the action of $\mathrm{Cent}(C)$ on G $(\bar{\theta})$ is defined as $\bar{\theta}(g,c)=\begin{cases}\theta(g,c) & \text{if } \theta(g,c)\\ c & \text{e.w.}\end{cases}$, for all $g\in G$ and $c\in \mathrm{Cent}(C)$ and $i:\mathrm{Cent}(C)\to C$ is an inclusion map with a trivial action.

(2) for any crossed module of groups (C,G,∂,θ), there is a bicrossed module of bigroups $\left(\mathrm{Cent}(C)\cup \mathrm{Cent}(\mathrm{Inn}(C)), G\cup \mathrm{Inn}(G), C\cup \mathrm{Inn}(C), \{\bar{\partial},\overline{\rho_2}\}, i\right)$ where the definitions of $\bar{\partial},\overline{\rho_2},i$ and the actions is are similar to the previous sections and the above example.

PROPOSITION 4.5. There are two functors $F:\mathrm{CmodBiGrps}\longrightarrow \mathrm{BiCmodBiGrps}$ and $T:\mathrm{BiCmodBiGrps}\longrightarrow \mathrm{CmodBiGrps}$.

Proof:- define $F:\mathrm{CmodBiGrps}\longrightarrow \mathrm{BiCmodBiGrps}$ as follows:

(i) $F(C,G,\partial)=\left(\mathrm{Cent}(C),G,C,\hat{\partial},i\right)$, for all $(C,G,\partial)\in \mathrm{obCmodBiGrps}$, and

(ii) $F(\mu,\eta)=\left(\tilde{\tilde{\mu}},\eta,\mu\right)$, for all $(\mu,\eta)\in \mathrm{Mor}_{\mathrm{CmodBiGrps}}\left((C,G,\partial),(\acute{C},\acute{G},\acute{\partial})\right)$.

Also, define $T:\mathrm{BiCmodBiGrps}\longrightarrow \mathrm{CmodBiGrps}$ as follows:

(i) $T\left((C,G,H,\partial,\delta)\right)=(C,C,I)$, for all $(C,G,H,\partial,\delta)\in \mathrm{obBiCmodGrps}$, and

(ii) $F(\mu,\eta,\omega)=(\mu,\mu)$,
for all $(\mu,\eta,\omega)\in \mathrm{Mor}_{\mathrm{BiCmodGrps}}\left((C,G,H,\partial,\delta),(\acute{C},\acute{G},\acute{H},\acute{\partial},\acute{\delta})\right)$.

5. Adjoint Pair of Functors

LEMMA 5.1. There are two covariant functors**;**

(1) The functor $F:\mathrm{BiCModBiGrps}\to \mathrm{BiGrps}$ is defined by:
(i) $F(C,G,H,\partial,\delta)=(C)$, for all $(C,G,H,\partial,\delta)\in \mathrm{ObBiCModBiGrps}$.
(ii) $L\left((\mu,\eta,\omega)\right)=\mu$,
for all $(\mu,\eta,\omega)\mathrm{Mor}_{\mathrm{BiCModBiGrps}}(C,G,H,\partial,\delta)\to\left(\acute{C},\acute{G},\acute{H},\acute{\partial},\acute{\delta}\right)$.
(2) $L:\mathrm{BiGrps}\to \mathrm{BiCModBiGrps}$ is defined by:
(i) $L(C)=(\mathrm{Cent}(C),C,C,i,i)$, for all $(C)\in \mathrm{ObBiGrps}$.
(ii) $L(\mu)=\left(\tilde{\tilde{\mu}},\mu,\mu\right)$, where $\hat{\mu}=\mu/\mathrm{Cent}(C)$ and $\tilde{\mu}:C\to \mathrm{Cent}\acute{C}$ are defined by; $\tilde{\mu}(c)=\begin{cases}\mu(c) & \text{if } \mu(c)\in \mathrm{Cent}(\acute{C})\\ 1_{\acute{C}} & \text{otherwise}\end{cases}$, for all $\mu\in \mathrm{Mor}_{\mathrm{BiGrps}}(C,\acute{C})$.

THEOREM 5.2. $L:\mathrm{BiGrps}\to \mathrm{BiCModBiGrps}$ is a left adjoint functor of $F:\mathrm{BiCModBiGrps}\to \mathrm{BiGrps}$.

Proof. We shall show that there is a natural isomorphism $\Phi:\mathrm{Mor}_{\mathrm{BiCModBiGrps}}(L-,-)\to \mathrm{Mor}_{\mathrm{BiGrps}}(-,F-)$, where $\mathrm{Mor}_{\mathrm{BiCModBiGrps}}(L-,-)$, $\mathrm{Mor}_{\mathrm{BiGrps}}(-,F-):\mathrm{BiGrps}^{\mathrm{op}}\times \mathrm{BiCModBiGrps}\to S$ are bifunctors ; the notation $\mathrm{BiGrps}^{\mathrm{op}}$ which denotes the opposite (or dual) category of

BiGrps, and S which is the category of sets, are defined respectively by the following compositions:

$$\mathrm{BiGrps}^{op} \times \mathrm{BiCModBiGrps} \xrightarrow{L^{op}\times I_{\mathrm{BiCModBiGrps}}} \mathrm{BiCModBiGrps}^{op} \times \mathrm{BiCModBiGrps} \xrightarrow{E_{\mathrm{CModComp}}} S$$

and $\mathrm{BiGrps}^{op} \times \mathrm{BiCModBiGrps} \xrightarrow{I_{\mathrm{BiGrps}^{op}}\times F} \mathrm{BiGrps}^{op} \times \mathrm{BiGrps} \xrightarrow{E_{\mathrm{BiGrps}}} S$.

Define a function $\Phi: \mathrm{Mor}_{\mathrm{BiCModBiGrps}}(L-,-) \to \mathrm{Mor}_{\mathrm{BiGrps}}(-,F-)$ as follows;

for all $A = \left(C^{op}, \left(\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta}\right)\right) \in \mathrm{Ob}(\mathrm{BiGrps}^{op} \times \mathrm{BiCModBiGrps})$,

$\Phi(A): \mathrm{Mor}_{\mathrm{BiCModBiGrps}}\left((\mathrm{Cent}(C), C, C, i, i), \left(\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta}\right)\right) \to$

$\mathrm{Mor}_{\mathrm{BiGrps}}\left(C, \acute{C}\right)$ is defined by $\Phi(A)\left((\mu, \eta, \omega)\right) = \breve{\mu}$, where $\breve{\mu}: C \to \acute{C}$ is defined as follows;

$$\breve{\mu}(c) = \begin{cases} \tau(c) & \text{if } \mu \text{ is defined already as a restriction of } \tau: C \to \acute{C} \text{ on } \mathrm{Cent}(C) \\ \breve{\mu}(c) & \text{otherwise} \end{cases}, \quad \text{for all}$$

$(\mu, \eta, \omega) \in \mathrm{Mor}_{\mathrm{BiCModBiGrps}}\left((\mathrm{Cent}(C), C, C, i, i), \left(\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta}\right)\right)$.

Firstly, we first show that Φ is a natural transformation. To do this,

let $\left(\mu^{op}, (\acute{\mu}, \acute{\eta}, \acute{\omega})\right) \in \mathrm{Mor}_{\mathrm{BiGrps}^{op}\times \mathrm{BiCModBiGrps}}(A, B)$, where $A = \left(C^{op}, \left(\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta}\right)\right)$ and $B = \left(D^{op}, \left(\acute{D}, \acute{E}, \acute{K}, \acute{\beta}, \acute{\alpha}\right)\right)$. It is enough to show the commutativity of the following diagram:

$$\begin{array}{cccc}
A & \mathrm{Mor}_{\mathrm{BiCModBiGrps}}\left((\mathrm{Cent}(C), C, C), \left(\acute{C}, \acute{G}, \acute{H}\right)\right) & \xrightarrow{\Phi(A)} & \mathrm{Mor}_{\mathrm{BiGrps}}\left(C, \acute{C}\right) \\
\downarrow \left(\mu^{op}, (\acute{\mu}, \acute{\eta}, \acute{\omega})\right) & \downarrow E_{\mathrm{BiCModBiGrps}}\left(\left(\hat{\tilde{\mu}}^{op}, \mu^{op}, \mu^{op}\right), (\acute{\mu}, \acute{\eta}, \acute{\omega})\right) & & \downarrow E_{\mathrm{BiGrps}}(\mu^{op}, \acute{\mu}) \\
B & \mathrm{Mor}_{\mathrm{BiCModBiGrps}}\left((\mathrm{Cent}(D), D, D), \left(\acute{D}, \acute{E}, \acute{K}\right)\right) & \xrightarrow{\Phi(B)} & \mathrm{Mor}_{\mathrm{BiGrps}}\left(D, \acute{D}\right)
\end{array}$$

Let $(\mu_1, \mu_2, \mu_3) \in \mathrm{Mor}_{\mathrm{BiCModBiGrps}}\left((\mathrm{Cent}(C), C, C), \left(\acute{C}, \acute{G}, \acute{H}\right)\right)$. Therefore

$$\begin{aligned}
\left(E_{\mathrm{BiGrps}}(\mu^{op}, \acute{\mu})\, \Phi(A)\right)(\mu_1, \mu_2, \mu_3) &= E_{\mathrm{BiGrps}}(\mu^{op}, \acute{\mu})\left(\Phi(A)(\mu_1, \mu_2, \mu_3)\right) \\
= E_{\mathrm{BiGrps}}(\mu^{op}, \acute{\mu})(\breve{\mu}_1) \;=\; \acute{\mu}\breve{\mu}_1\mu = \acute{\mu}\mu_1\tilde{\mu} &= \acute{\mu}\mu_1\breve{\hat{\tilde{\mu}}} = \left(\widetilde{\acute{\mu}\mu_1\hat{\tilde{\mu}}}\right) \\
&= \Phi(B)\left(\acute{\mu}\mu_1\hat{\tilde{\mu}}, \acute{\eta}\mu_2\mu, \acute{\omega}\mu_3\mu\right) \\
&= \Phi(B)\left((\acute{\mu}, \acute{\eta}, \acute{\omega})(\mu_1, \mu_2, \mu_3)\left(\hat{\tilde{\mu}}, \mu, \mu\right)\right) \\
&= E_{\mathrm{BiCModBiGrps}}\left(\left(\hat{\tilde{\mu}}^{op}, \mu^{op}, \mu^{op}\right), (\acute{\mu}, \acute{\eta}, \acute{\omega})\right)(\mu_1, \mu_2, \mu_3).
\end{aligned}$$

Also, define a function $\Psi: \mathrm{Mor}_{\mathrm{BiGrps}}(-,F-) \to \mathrm{Mor}_{\mathrm{BiCModBiGrps}}(L-,-)$ as follows;

for all $C = \left(C^{op}, \left(\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta}\right)\right) \in \mathrm{Ob}(\mathrm{BiGrps}^{op} \times \mathrm{BiCModBiGrps})$,

$\Psi(C): \mathrm{Mor}_{\mathrm{BiGrps}}\left(C, \acute{C}\right) \to$

$$\mathrm{Mor}_{\mathrm{BiCModBiGrps}}\left((\mathrm{Cent}(C), C, C, i, i), \left(\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta}\right)\right)$$

is defined by $\Psi(C)(\mu) = \left(\hat{\mu}, \acute{\partial}\mu, \acute{\delta}\mu\right)$,(where $\hat{\mu} = \mu/\mathrm{Cent}(C)$), for all

$\mu \in Mor_{BiGrps}(C, \acute{C})$. We turn now to show that Ψ is a natural transformation.
Let $(\mu^{op}, (\acute{\mu}, \acute{\eta}, \acute{\omega})) \in Mor_{BiGrps^{op} \times BiCModBiGrps}(C, D)$, where $C = \left(C^{op}, (\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta})\right)$ and $D = \left(D^{op}, (\acute{D}, \acute{E}, \acute{K}, \acute{\beta}, \acute{\alpha})\right)$. It is enough to show the commutativity of the following diagram:

$$\begin{array}{ccccc} C & Mor_{BiGrps}(C, \acute{C}) & \xrightarrow{\Psi(C)} & Mor_{BiCModBiGrps}\left((Cent(C), C, C), (\acute{C}, \acute{G}, \acute{H})\right) \\ \downarrow (\mu^{op}, (\acute{\mu}, \acute{\eta}, \acute{\omega})) & \downarrow E_{BiGrps}(\mu^{op}, \acute{\mu}) & & \downarrow E_{BiCModBiGrps}\left((\hat{\tilde{\mu}}^{op}, \mu^{op}, \mu^{op}), (\acute{\mu}, \acute{\eta}, \acute{\omega})\right) \\ D & Mor_{BiGrps}(D, \acute{D}) & \xrightarrow{\Psi(D)} & Mor_{BiCModBiGrps}\left((Cent(D), D, D), (\acute{D}, \acute{E}, \acute{K})\right) \end{array}$$

Let $\mu_1 \in Mor_{BiGrps}(C, \acute{C})$. Therefore

$\left(E_{BiCModBiGrps}\left((\hat{\tilde{\mu}}^{op}, \mu^{op}, \mu^{op}), (\acute{\mu}, \acute{\eta}, \acute{\omega})\right)\Psi(C)\right)(\mu_1) =$
$E_{BiCModBiGrps}\left((\hat{\tilde{\mu}}^{op}, \mu^{op}, \mu^{op}), (\acute{\mu}, \acute{\eta}, \acute{\omega})\right)(\hat{\mu}_1, \acute{\partial}\mu_1, \acute{\delta}\mu_1) =$
$(\acute{\mu}, \acute{\eta}, \acute{\omega})(\hat{\mu}_1, \acute{\partial}\mu_1, \acute{\delta}\mu_1)(\hat{\tilde{\mu}}, \mu, \mu) = (\acute{\mu}\hat{\mu}_1\hat{\tilde{\mu}}, \acute{\eta}(\acute{\partial}\mu_1)\mu, \acute{\omega}(\acute{\delta}\mu_1)\mu) =$
$\left(\acute{\mu}\breve{\tilde{\mu}}_1\hat{\mu}, (\acute{\eta}\,\acute{\partial})(\mu_1\mu), (\acute{\omega}\acute{\delta})(\mu_1\mu)\right) = \left(\acute{\mu}\mu_1\hat{\mu}, (\acute{\beta}\acute{\mu})(\mu_1\mu), (\acute{\alpha}\acute{\mu})(\mu_1\mu)\right) =$
$\left(\widehat{\acute{\mu}\mu_1\mu}, \acute{\beta}(\acute{\mu}\mu_1\mu), \acute{\alpha}(\acute{\mu}\mu_1\mu)\right) = \Psi(D)(\acute{\mu}\mu_1\mu) = \Psi(D)\left(E_{BiGrps}(\mu^{op}, \acute{\mu})(\mu_1)\right) =$
$\Psi(D)E_{BiGrps}(\mu^{op}, \acute{\mu})(\mu_1)$.

To proceed the proof, we need only to show that $\Psi\Phi = I_{Mor_{BiCModBiGrps}(L-,-)}$ and $\Phi\Psi = I_{Mor_{BiGrps}(-,F-)}$. Let $C = \left(C^{op}, (\acute{C}, \acute{G}, \acute{H}, \acute{\partial}, \acute{\delta})\right) \in Ob(BiGrps^{op} \times BiCModBiGrps)$. We need to prove the commutativity of the following diagram:

$$\begin{array}{ccc} Mor_{BiGrps}(C, \acute{C}) & \xrightarrow{\Psi(C)} & Mor_{BiCModBiGrps}\left((Cent(C), C, C), (\acute{C}, \acute{G}, \acute{H})\right) \\ & \searrow I_{Mor_{BiGrps}(-,F-)}(C) & \downarrow \Phi(C) \\ & & Mor_{BiGrps}(C, \acute{C}) \end{array}$$

Let $\mu \in Mor_{BiGrps}(C, \acute{C})$. Therefore

$$(\Phi(C)\Psi(C))(\mu) = \Phi(C)(\hat{\mu}, \acute{\partial}\mu, \acute{\delta}\mu) = \mu = I_{Mor_{BiGrps}(C_*, \mu_*), (\acute{C}_*, \acute{\mu}_*)}(\mu)$$
$$= \left(I_{Mor_{BiCModBiGrps}(-,F-)}(A)\right)(\mu).$$

Likewise, $\Psi\Phi = I_{Mor_{BiCModBiGrps}(L-,-)}$. Hence L is a left adjoint functor of F. ■

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الموديولات المتصالبة لزمر الثنائية والموديولات المتصالبة الثنائية لزمر الثنائية

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الخلاصة

يعنى هذا البحث بتقديم مفهومي الموديولات المتصالبة لزمر الثنائية والموديولات المتصالبة الثنائية لزمر الثنائية كتعميم مناسب لمفهومي الموديولات المتصالبة لزمر والموديولات المتصالبة الثنائية لزمر, على التوالي. عمل هذا بغمر فصيلتي الموديولات المتصالبة لزمر والموديولات المتصالبة الثنائية لزمر في فصيلتي الموديولات المتصالبة لزمر الثنائية والموديولات المتصالبة الثنائية لزمر الثنائية (وفقا للتشاكل التقابلي للفصائل). مفهوم الزمرة الموسعة والتشاكل الزمري الموسع كذلك قدم في اول البحث. أخيرا, قدم زوج مترافق من المقرنات التغايرية بين فصيلة الموديولات المتصالبة الثنائية لزمر الثنائية وفصيلة الزمر الثنائية.